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THREE-DIMENSIONAL CORRECTIONS FOR A PLANE STRESS PROBLEM

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Abstract—A generalized theory of plane stress is applied to the particular problem of a semi-infinite, isotropically elastic sheet of uniform thickness with a sinusoidally varying normal edge load (parallel to the plane of the sheet). Approximate three-dimensional corrections to the elementary two-dimensional plane stress solution are obtained which are roughly proportional to Poisson's ratio and which result in a maximum stress which is 10–20% larger for Poisson's ratio varying from 0.3 to 0.5.

1. INTRODUCTION

The purpose of this paper is to present an application of a theory of generalized plane stress which was first proposed by Reissner in 1943 [1] and was recently rederived in somewhat more general form [2]. The particular problem presented here was first considered by the author in an unpublished Master's thesis [3] written under the direction of Professor Reissner and completed in February 1946. It is most fitting for the current version of the problem to appear in a collection honoring Professor Reissner's seventieth anniversary.

The problem is that of a semi-infinite elastic sheet or layer of uniform thickness with a normal edge load, parallel to the plane of the sheet, which varies sinusoidally along the edge of the sheet but is uniform across the thickness (see Fig. 1). The elementary two-dimensional plane stress solution of the problem for an isotropic layer is independent of Poisson's ratio ν and neglects totally the transverse shear and normal stresses. The three-dimensional corrections obtained here show that the elementary theory is indeed valid for practical purposes for small thickness to load wavelength ratios but that, when the load wavelength is comparable to the thickness, the maximum stress at the edge of the sheet may be up to 20% larger (for $\nu = \frac{1}{2}$) than the value given by the elementary theory. Also, the transverse normal and shear stresses may be quite significant.

The theory applied here assumes that stresses vary across the plate thickness in the simplest way compatible with the equilibrium equations and this assumed thickness variation becomes less accurate as the load wavelength becomes significantly smaller than the sheet thickness. Otherwise, there would be a transition to plane strain theory. As the load wavelength approaches zero, the limiting value of the transverse normal stress obtained here exceeds the plane strain value by 31%, showing that the present theory is not quantitatively correct at such extremes. However, it does possess the right qualitative features, as all other stress quantities do approach the values given by plane strain theory in the middle portion of the layer.

2. EQUATIONS FOR GENERALIZED PLANE STRESS

The theory used here assumes that the three-dimensional stresses of linear elasticity theory have the approximate form,

$$(\sigma_x, \sigma_y, \tau_{xy}) = \frac{(N_{xx}, N_{yy}, N_{xy})}{2c} + \frac{(R_{xx}, R_{yy}, R_{xy})}{2c} Z''(z) \tag{1}$$

$$(\tau_{xz}, \tau_{yz}) = \frac{(S_x, S_y)}{2c} Z'(z), \quad \sigma_z = -\frac{T}{2c} Z(z), \tag{2}$$

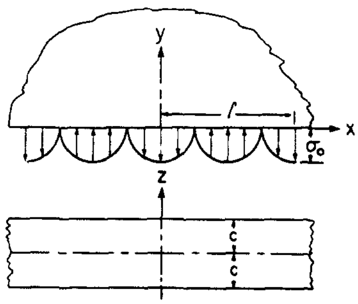

Fig. 1. Semi-infinite elastic sheet of thickness $2c$ with a sinusoidal normal edge load. $\sigma_y = \sigma_0 \cos \alpha x$, $\alpha = 2\pi/l$.

where

$$Z(z) = -\frac{c^2}{4}\left(1 - \frac{z^2}{c^2}\right)^2, \quad Z'(z) = c \cdot \frac{z}{c}\left(1 - \frac{z^2}{c^2}\right), \quad Z''(z) = 1 - 3\frac{z^2}{c^2}. \tag{3}$$

The form assumed is consistent with the three-dimensional equilibrium equations and with the traction conditions $\tau_{xz} = \tau_{yz} = \sigma_z = 0$ on the faces, $z = \pm c$, of the sheet. The form also corresponds to the assumption that in-plane components of stress (as well as edge loads) are symmetric with respect to the middle plane, $z = 0$. The stress quantities, $N_{xx}, \ldots, T$, are functions of the in-plane coordinates x, y. Equilibrium equations for the stress quantities and their relation to weighted displacement component averages over the sheet thickness are obtained from a variational principle in [2]. A tenth-order system of differential equations is found which may be reduced to the following three uncoupled partial differential equations for three stress functions, φ, ψ, Ω:

$$\nabla^4\varphi = 0, \quad A_0 c^2 \nabla^2\Omega - \Omega = 0, \tag{4a,b}$$

$$A_1 c^4 \nabla^4 \psi + A_2 c^2 \nabla^2\psi + \psi = 0. \tag{5}$$

In this, ∇^2 denotes the two-dimensional Laplace operator and the coefficients A_i are dimensionless functions of the elastic constants, which are determined in [2] for a homogeneous transversely isotropic layer. Here, only the completely isotropic case is considered. The coefficients are then given by

$$A_0 = \frac{2}{21}, \quad A_1 = \frac{2}{63}\left(\frac{1 - 69\nu^2/70}{1 - \nu^2}\right), \quad A_2 = -\frac{4}{21}. \tag{6}$$

Only one coefficient depends upon Poisson's ratio ν and even it may be treated as independent of ν for practical purposes. In this case ψ, as well as Ω, will depend upon ν only through the boundary conditions.

Once eqns (4) and (5) have been solved, subject to appropriate boundary conditions, the various stress quantities are given in terms of two auxiliary functions,

$$K = \varphi - \frac{2\nu}{15} c^2 \psi, \tag{7}$$

$$\chi = \frac{2}{21}\left[(2 - \nu)\psi - \frac{(1 - (69/70)\nu^2)}{3(1 + \nu)} c^2\nabla^2\psi\right] - \frac{\nu}{6(1 + \nu)} \nabla^2 \varphi, \tag{8}$$

by the following formulas:

$$N_{xx} = K_{,yy}, \quad N_{yy} = K_{,xx}, \quad N_{xy} = -K_{,xy}, \tag{9}$$

$$R_{xx} = \psi - c^2 \chi_{,yy} + (4c^2/21)\, \Omega_{,xy}, \tag{10}$$

$$R_{yy} = \psi - c^2 \chi_{,xx} - (4c^2/21)\, \Omega_{,xy}, \tag{11}$$

$$R_{xy} = c^2 \chi_{,xy} + (2c^2/21)\, \Omega_{,xx}, \tag{12}$$

$$S_x = -\psi_{,x} - \Omega_{,y}, \quad S_y = -\psi_{,y} + \Omega_{,x}, \quad T = -\nabla^2 \psi. \tag{13a,b,c}$$

Various features of the above formulas and differential equations (4) and (5) may be pointed out. First, if one formally puts the half-thickness $c = 0$, then, from eqns (4b) and (5), $\Omega = 0$, $\psi = 0$, all quantities except N_{xx}, N_{yy}, N_{xy} vanish, and $K = \varphi$ is a biharmonic function which may be identified with the Airy stress-function of elementary plane stress theory, so that the present theory reduces to the elementary theory. If $\psi \equiv 0$ and $\Omega \equiv 0$ with $c \neq 0$, then the formulas yield expressions for $R_{xx}, \ldots, R_{xy}$ in terms of φ from the last term in eqn (8). These are equivalent to the correction terms given by an exact solution of three-dimensional elasticity theory (see Love [4], Section 145, or Timoshenko and Goodier [5], Section 84). However, this solution implies that a parabolic variation of σ_x, σ_y, τ_{xy} in the thickness coordinate z holds throughout the layer, even at the edges. The terms in ψ and Ω represent edge-zone corrections which are important only over distances, into the interior of a sheet, of the order of the thickness $2c$, while terms in φ represent the interior state of stress. The function ψ represents transverse normal stress and shear stress effects, while Ω, as is shown in [2], represents transverse shear effects only, or the shearing action in layers parallel to the plane of the sheet.

3. SEMI-INFINITE SHEET, SINUSOIDAL LOAD

Consider a semi-infinite sheet occupying the region, $y \geq 0$, $-c \leq z \leq c$ (see Fig. 1), with the faces $z = \pm c$ free of stress and with edge stresses given, for $-\infty < x < \infty$, by

$$\sigma_y(x, 0, z) = \sigma_0 \cos(\alpha x), \quad \tau_{xy}(x, 0, z) = 0, \quad \tau_{xz}(x, 0, z) = 0 \tag{14a,b,c}$$

where σ_0 is a constant and $\alpha = 2\pi/l$. As $y \to \infty$, all stress quantities are to approach zero.

An approximate solution of this three-dimensional problem is given by the formulas of the previous section if, at the edge $y = 0$, the following five conditions hold (with $N_0 = 2c\sigma_0$):

$$N_{yy}(x, 0) = N_0 \cos(\alpha x), \; N_{xy}(x, 0) = 0, \tag{15a,b}$$

$$R_{yy}(x, 0) = 0, \quad R_{xy}(x, 0) = 0, \quad S_y(x, 0) = 0. \tag{16a,b,c}$$

Again, all quantities are to vanish at $y = \infty$.

From the form of the boundary conditions and the formulas for the stress quantities, one may anticipate that the functions φ and ψ are proportional to $\cos \alpha x$ and the function Ω to $\sin \alpha x$. Substituting

$$\varphi(x, y) = f(y) \cos \alpha x \tag{17}$$

into eqn (4a) yields a fourth-order ordinary differential for $f(y)$ with a general solution,

$$f(y) = (c_1 + c_2 y)e^{-\alpha y} + (c_3 + c_4 y)\, e^{\alpha y}. \tag{18}$$

The conditions at $y = \infty$ require that $c_3 = c_4 = 0$. Anticipating subsequent formulas, it is convenient to write

$$\varphi(x, y) = -(N_0/\alpha^2)[1 + C_1 + (1 + C_2)\alpha y]\, e^{-\alpha y} \cos \alpha x. \tag{19}$$

If C_1 and C_2 are set equal to zero in eqn (19), one has the Airy stress-function of elementary plane stress which corresponds to the stress quantities,

$$N_{xx}^{(0)} = N_0(1 - \alpha y)e^{-\alpha y}\cos\alpha x, \tag{20}$$

$$N_{yy}^{(0)} = N_0(1 + \alpha y)e^{-\alpha y}\cos\alpha x, \quad N_{xy}^{(0)} = N_0\alpha y e^{-\alpha y}\cos\alpha x. \tag{21a,b}$$

The above formulas (with all other quantities equal to zero) also correspond to the exact solution of the problem when Poisson's ratio ν vanishes.

The differential operator in eqn (5) for ψ turns out to have complex-valued factors. The equation may be written in the form,

$$(c^2\nabla^2 - \mu^2)(c^2\nabla^2 - \bar{\mu}^2)\psi = 0, \tag{5'}$$

where superposed bars denote complex conjugates and where

$$\left(1 - \frac{69}{70}\nu^2\right)\mu^4 - 6(1 - \nu^2)\mu^2 + \frac{63}{2}(1 - \nu^2) = 0. \tag{22}$$

The solution of eqn (22) may be written as

$$\mu^2 = \frac{3(1 - \nu^2)}{1 - 69\nu^2/70}\left[1 + \frac{i}{2}\sqrt{\left(10 + \frac{\nu^2}{5(1 - \nu^2)}\right)}\right] \approx 3\left(1 + i\frac{\sqrt{10}}{2}\right). \tag{23}$$

With ψ proportional to $\cos\alpha x$, the general solution of eqn (5′) which vanishes at $y = \infty$ then takes the form,

$$\psi(x, y) = N_0(B\,e^{-\beta y} + \bar{B}\,e^{-\bar{\beta}y})\cos\alpha x, \tag{24}$$

where

$$(c\beta)^2 = (c\alpha)^2 + \mu^2. \tag{25}$$

Finally, the appropriate solution of eqn (4b), with $A_0 = 2/21$, is given by

$$\Omega(x, y) = N_0C_0\,e^{-\gamma y}\sin\alpha x, \tag{26}$$

where

$$(c\gamma)^2 = (c\alpha)^2 + 21/2. \tag{27}$$

Substituting the above formulas for φ, ψ and Ω into eqns (7)–(13) results in the following expressions for the various stress quantities:

$$N_{xx}/N_0 = \{[1 - C_1 + 2C_2 - (1 + C_2)\,\alpha y]\,e^{-\alpha y} - (2\nu/15)[(c\beta)^2\,Be^{-\beta y} + (c\bar{\beta})^2\,\bar{B}\,e^{-\bar{\beta}y}]\}\cos\alpha x, \tag{28}$$

$$N_{yy}/N_0 = \{[1 + C_1 + (1 + C_2)\alpha y]e^{-\alpha y} + (2\nu/15)(c\alpha)^2(Be^{-\beta y} + \bar{B}\,e^{-\bar{\beta}y})\}\cos\alpha x, \tag{29}$$

$$N_{xy}/N_0 = \{C_1 - C_2 + (1 + C_2)\alpha y)e^{-\alpha y} + (2\nu/15)c\alpha\,(c\beta Be^{-\beta y} + c\bar{\beta}\,\bar{B}e^{-\bar{\beta}y})\}\sin\alpha x, \tag{30}$$

$$R_{xx}/N_0 = \Big\{[1 - (c\beta)^2 M]Be^{-\beta y} + [1 - (c\bar{\beta})^2 \bar{M}]\bar{B}e^{-\bar{\beta} y} + \frac{\nu(c\alpha)^2}{3(1+\nu)}(1 + C_2)e^{-\alpha y} - \frac{4c^2\alpha\gamma}{21} C_0\, e^{-\gamma y}\Big\} \cos \alpha x, \quad (31)$$

$$R_{yy}/N_0 = \Big\{[1 + (c\alpha)^2 M]Be^{-\beta y} + [1 + (c\alpha)^2 \bar{M}]\bar{B}e^{-\bar{\beta} y} - \frac{\nu(c\alpha)^2}{3(1+\nu)}(1 + C_2)\, e^{-\alpha y} + \frac{4c^2\alpha\gamma}{21} C_0\, e^{-\gamma y}\Big\} \cos \alpha x, \quad (32)$$

$$R_{xy}/N_0 = \Big\{c\alpha\,(c\beta M B e^{-\beta y} + c\bar{\beta}\bar{M}\bar{B}e^{-\bar{\beta} y}) - \frac{\nu(c\alpha)^2}{3(1+\nu)}(1 + C_2)e^{-\alpha y} + \frac{2}{21}[(c\gamma)^2 + (c\alpha)^2]C_0 e^{-\gamma y}\Big\} \sin \alpha x, \quad (33)$$

$$cS_x/N_0 = [c\alpha(Be^{-\beta y} + \bar{B}e^{-\bar{\beta} y}) + c\gamma\, C_0 e^{-\gamma y}] \sin \alpha x, \quad (34)$$

$$cS_y/N_0 = [c\beta B e^{-\beta y} + c\bar{\beta}\bar{B}e^{-\beta y} + c\alpha\, C_0 e^{-\gamma y}] \cos \alpha x, \quad (35)$$

$$c^2 T/N_0 = [-\mu^2 B e^{-\beta y} - \bar{\mu}^2\bar{B}e^{-\bar{\beta} y}] \cos \alpha x, \quad (36)$$

where

$$M = \frac{2}{21}\left[2 - \nu - \frac{1 - (69/70)\nu^2}{3(1+\nu)}\mu^2\right] = \frac{2\nu}{21} + \frac{1-\nu}{\mu^2} = \frac{2}{21} - i\frac{(1-\nu)}{21}\sqrt{\left(10 + \frac{\nu^2}{5(1-\nu^2)}\right)}. \quad (37)$$

Applying the boundary conditions, eqns (15) and (16), to the above expressions, one obtains the following five linear algebraic equations for the integration constants, C_0, C_1, C_2, B, $\bar{B}$:

$$C_1 + \frac{2\nu}{15}(c\alpha)^2(B + \bar{B}) = 0, \quad (38)$$

$$C_1 - C_2 + \frac{2\nu}{15} c\alpha\,(c\beta B + c\bar{\beta}\bar{B}) = 0, \quad (39)$$

$$\frac{4c^2}{21}\alpha\gamma\, C_0 - \frac{\nu(c\alpha)^2}{3(1+\nu)} C_2 + [1 + (c\alpha)^2 M]B + [1 + (c\alpha)^2\bar{M}]\bar{B} = \frac{\nu(c\alpha)^2}{3(1+\nu)}, \quad (40)$$

$$\frac{2c^2}{21}(\gamma^2 + \alpha^2)C_0 - \frac{\nu(c\alpha)^2}{3(1+\nu)} C_2 + c^2\alpha\beta M\beta + c^2\alpha\bar{\beta}\bar{M}\bar{B} = \frac{\nu(c\alpha)^2}{3(1+\nu)}, \quad (41)$$

$$c\alpha C_0 + c\beta B + c\bar{\beta}\bar{B} = 0. \quad (42)$$

By decomposing complex-valued quantities into real and imaginary parts, writing

$$\beta = \beta_r + i\,\beta_i, \quad B = B_r + i\,B_i, \quad M = M_r + i\,M_i, \quad (43)$$

the above equations may be reduced to equations for real-valued quantities and solved numerically on a computer for various values of $c\alpha = 2\pi c/l$ and ν. Table 1 lists some

TABLE 1

ν = .5								
cα	$c\beta_r$	$c\beta_i$	cγ	C_0	C_1	C_2	$2B_r$	$2B_i$
.5	2.1180	1.1182	3.2787	.02266	-.00032	-.00070	.01934	.04676
1.0	2.2557	1.0499	3.3912	.07096	-.00314	-.00787	.04715	.16889
2.0	2.7772	.8528	3.8079	.15511	-.00978	-.05115	.03669	.48328
3.0	3.5266	.6716	4.4159	.18462	.01367	-.09710	-.02279	.70506
5.0	5.3089	.4461	5.9582	.18524	.16685	-.14188	-.10011	.88480
10.0	10.1509	.2333	10.5119	.17347	.99016	-.16633	-.14852	.97331

of the values obtained for $\nu = \frac{1}{2}$. The results, substituted into eqns (28)–(36), determine the various stress quantities. Since all such quantities decay exponentially as y increases, maximum values of the quantities not prescribed in the boundary conditions occur at $y = 0$. After some algebraic reductions, using the relations (22), (25) and (38)–(42), one may deduce the following formulas:

$$c^2T(0, 0)/N_0 = -(\mu^2B + \bar{\mu}^2\bar{B}) = -2[\mathrm{Re}(\mu^2)B_r - \mathrm{Im}(\mu^2)B_i], \tag{44}$$

$$N_{xx}(0, 0)/N_0 = 1 + 2C_2 + (2\nu/15)c^2T(0, 0)/N_0, \tag{45}$$

$$R_{xx}(0, 0)/N_0 = 2(1 + \nu)B_r + (2\nu/21)c^2T(0, 0)/N_0, \tag{46}$$

$$cS_x(\pi/2\alpha, 0)/N_0 = 2c\alpha B_r + c\gamma C_0. \tag{47}$$

Approximations for the maximum values of the three-dimensional stresses may be obtained from eqns (1) and (2) taking (3) into account. One has

$$\max \sigma_x = \sigma_x(0, 0, 0) = [N_{xx}(0, 0) + R_{xx}(0, 0)]/(2c), \tag{48}$$

$$\max \tau_{xz} = \tau_{xz}(\pi/(2\alpha), 0, c/\sqrt{3}) = (2\sqrt{3}/9)cS_x(\pi/(2\alpha), 0)/(2c), \tag{49}$$

$$\max \sigma_z = \sigma_z(0, 0, 0) = c^2T(0, 0)/(8c). \tag{50}$$

Also, the minimum edge value of σ_x occurs at $z = \pm c$ and is given by

$$\min \sigma_x = \sigma_x(0, 0, c) = [N_{xx}(0, 0) - 2R_{xx}(0, 0)]/(2c). \tag{51}$$

With $N_0 = 2c\,\sigma_0$, the ratios, $\max \sigma_x/\sigma_0$ and $\min \sigma_x/\sigma_0$, are graphed in Fig. 2 for

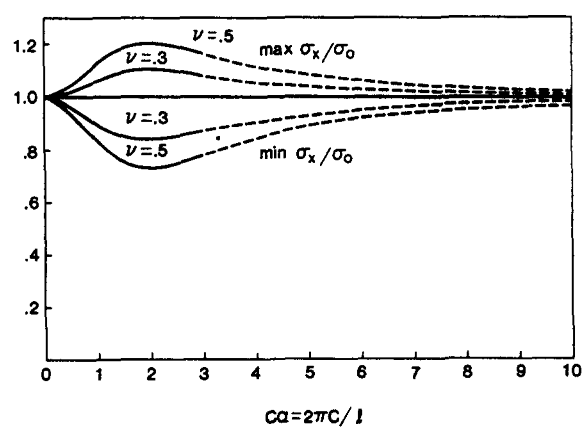

Fig. 2. Maximum and minimum values of σ_x/σ_0 at the edge, $y = 0$.

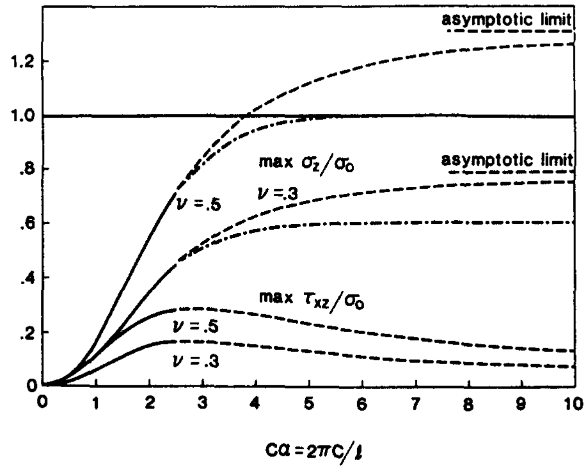

Fig. 3. Maximum values of σ_z/σ_0 and τ_{xz}/σ_0 at the edge, $y = 0$. ---- calculated, —·—·— conjectured.

different values of ν as functions of the ratio, $c\alpha = 2\pi c/l$. The elementary plane stress or plane strain edge value of σ_x/σ_0 is 1 and it is seen that this value is approached as $c\alpha \to 0$ or as $c\alpha \to \infty$. The greatest three-dimensional correction to the elementary two-dimensional theory occurs for $c\alpha \approx 2$ or $2c/l \approx 2/\pi$; that is, when the wavelength l of the load variation is comparable to the sheet thickness $2c$.

Graphs of max τ_{xz}/σ_0 and max σ_z/σ_0, which are neglected in the elementary theory, are shown in Fig. 3. The greatest value for τ_{xz} occurs for $c\alpha \approx 3$, so that this correction is important for even smaller wavelengths. The plane strain value of σ_z is $2\nu\sigma_0$ and the dash–dot curves in Fig. 3 show what the behavior of σ_z should be. As one approaches a plane strain loading approximation, it is known that there is a very narrow zone near the face of a thick layer where σ_z has a rapid transition to zero. Evidently, the assumed variation $Z(z)$ does not approximate this transition closely enough in the extreme of a very thick layer to give an accurate limiting value for σ_z. Limiting values of the solution of the linear system, (38)–(42), have been obtained analytically as $c\alpha \to \infty$. From these

TABLE 2

$\nu = .5$						
$c\alpha$	max N_{xx}/N_0	max R_{xx}/N_0	max σ_x/σ_0	min σ_x/σ_0	max τ_{xz}/σ_0	max σ_z/σ_0
.5	1.0095	.03681	1.0463	.9359	.0323	.0409
1.0	1.0282	.10212	1.1303	.8240	.1108	.1648
2.0	1.0430	.15883	1.2018	.7254	.2556	.5449
3.0	1.0330	.12808	1.1611	.7768	.2875	.8519
5.0	1.0156	.06364	1.0792	.8883	.2321	1.1225
10.0	1.0043	.01787	1.0221	.9685	.1302	1.2634
$\nu = .3$						
$c\alpha$	max N_{xx}/N_0	max R_{xx}/N_0	max σ_x/σ_0	min σ_x/σ_0	max τ_{xz}/σ_0	max σ_z/σ_0
.5	1.0039	.0208	1.0247	.9623	.0289	.0280
1.0	1.0114	.0572	1.0686	.8969	.0678	.1091
2.0	1.0168	.0889	1.1057	.8391	.1475	.3428
3.0	1.0128	.0720	1.0847	.8688	.1625	.5236
5.0	1.0060	.0358	1.0418	.9343	.1297	.6797
10.0	1.0016	.0100	1.0117	.9816	.0723	.7598

values, the result for σ_z is given by

$$\lim_{c\alpha \to \infty} \sigma_z(0, 0, 0) = \frac{21}{8} \nu\sigma_0 = (1.3125)2\nu\sigma_0. \tag{52}$$

Since the value of max σ_z, and therefore $T(0, 0)$, is clearly in error for $c\alpha > 2$ or 3, other quantities as given by eqns (44)–(47), are also probably inaccurate for $c\alpha > 2$, and these portions of the graphs in Figs. 2 and 3 have been drawn as dashed curves to indicate this. It is hoped that a further investigation will help to establish the range of validity of the result presented here as the ratio $c\alpha$ increases.

As $c\alpha \to 0$, the constants in the system, eqns (38)–(42), all approach zero at least as fast as $(c\alpha)^2$, showing that the error in the elementary plane stress solution is quadratic in $c\alpha$.

More extensive calculations were carried out than appear in Table 2, which list only some representative values. Results for $\nu = 0.2$ and 0.4 fit in between those for $\nu =$ 0, 0.3, 0.5. The variation in ν is approximately, but not exactly, linear. Finally, the computations were done using values of μ^2 and M as defined by eqns (23) and (37). Earlier computations [3] were done for the case $\nu = \frac{1}{2}$ using the approximation for μ^2 given in eqn (23) and a corresponding approximation for M. The earlier results agree, to within three significant figures at most points, with results presented here.

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